On-shell recursion relations using soft behaviour

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After briefly reviewing BCFW on-shell recursion method and Adler's zero, the behaviour of amplitudes containing Goldstone bosons, the construction of [1] which extends the on-shell recursion method is reviewed.

1 Introduction and Preliminaries

BCFW on-shell recursion[2, 3] is a method of constructing higher point tree level amplitudes from lower point tree level amplitudes recursively. Unlike traditional diagrammatic techniques where number of terms to sum up increases factorially in numbers of external legs, the recursive construction method only requires to sum over number of terms that grows linearly with number of external legs. This drastically reduces the computation costs required to compute an amplitude.

Another interesting property of this method is that lowest tree amplitudes—the three-point amplitudes—are specified by gauge invariance and symmetry, so the theory is predetermined from those considerations alone. For example, tree level gluon scattering amplitudes in four dimensional pure Yang-Mills theory are specified fully by Lorentz invariance and gauge invariance. In this respect, the recursive structure of amplitudes can be used to bootstrap the possible quantum field theories[4]. An exemplary case is Yang-Mills theory, which appears as the unique solution to a theory that is constructible by on-shell recursion relations obeying locality [5].

Unfortunately, the BCFW on-shell recursion method works in a rather limited scope; although gauge theories and graviton scattering amplitudes fall in the working category, theories such as nonlinear sigma models(NLSM), Dirac-Born-Infeld theory(DBI), and Galileon theories fall in the other category where this construction is invalid. This is because the amplitudes in such theories do not shrink fast enough at large momenta. However, this does not mean that recursive construction method does not exist for such theories. In fact, the on-shell amplitudes for them can be constructed recursively by exploiting soft behaviour called Adler's zero[1]. This recursive structure can be used to nail down all possible tree amplitudes of nontrivial effective scalar theories, thus providing an effective field theory equivalent of bootstrapping Yang-Mills theory from recursion relations[6]. The construction was extended in [7] to include scattering amplitudes of Goldstone bosons of broken symmetries, which become massive due to breaking of the symmetry they originate from. Since literature on on-shell recursion relations for massive particles are rather scarce, it would be an interesting research topic to see whether the new recursive method can provide more useful recursive relations for massive particle scattering amplitudes as well.

The manuscript is intended as an introduction and a short review on recursive construction of on-shell tree amplitudes for scalar theories suggested by [1]. The manuscript will explain briefly about a specific structure of momentum space correlation functions called pology, and derive BCFW recursion relations from it under suitable assumptions. After reviewing the soft behaviour of scattering amplitudes called Adler's zero, it will be outlined how the authors of [1] exploited it to build recursion relations for nontrivial effective scalar theories.

1.1 Polology

Polology refers to the pole structure of correlation functions in momentum space. Let us write n-point time ordered momentum space correlation function as follows.

$$G(q_1, q_2, \cdots, q_n) = \langle T\Phi_1(q_1)\Phi_2(q_2)\cdots\Phi_n(q_n)\rangle \tag{1}$$

Define $q = \sum_{i=1}^{r} q_i = -\sum_{i=r+1}^{n} q_i$, $1 \leq r \leq n$, and consider G as a function of q. The results of polology states that G contains the following pole structure.

$$G(q^2) = \frac{R}{q^2 + m^2 - i\epsilon} + \cdots \tag{2}$$

The numerator of the pole is given by the following formula.

$$R = -i(2\pi)^d\delta(\sum_1^n q_i)\sum_\sigma M_{0|q,\sigma}(q_1, \cdots, q_r)M_{q,\sigma|0}(q_{r+1}, \cdots, q_n) \tag{3}$$

$$(2\pi)^d\delta(\sum_1^r q_i)M_{0|q,\sigma}(q_1, \cdots, q_r) = \langle \text{vac}|\, T\Phi_1(q_1)\cdots\Phi_r(q_r)\,|\sigma(q)\rangle \tag{4}$$

$$(2\pi)^d\delta(\sum_{r+1}^n q_i)M_{q,\sigma|0}(q_{r+1}, \cdots, q_n) = \langle\sigma(q)|\, T\Phi_{r+1}(q_{r+1})\cdots\Phi_r(q_r)\,|\text{vac}\rangle \tag{5}$$

The index σ refers to the intermediate one-particle state of square mass m^2. It is important to note that one-particle state labeled by σ need not be a one-particle state obtained from quantisation of fundamental fields; it can be a composite one-particle state as well. The form (2) has the obvious interpretation as 'the contribution from a one particle exchange between the set of external legs 1, $\cdots$, r and the set of external legs $r+1$, $\cdots$, n', i.e. figure 1.

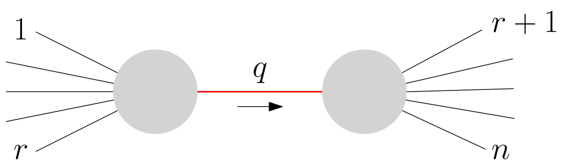

그림 1: Diagrammatic expression for (2).

Polology can be derived by isolating the contribution that can be interpreted as a single intermediate particle exchange, as the formula already suggests. Since the result is intuitively clear, the proof will be omitted. For a detailed derivation, consult chapter 10.2 of [8]

1.2 BCFW on-shell recursion

Consider a general n-point tree amplitude $A_n[p_i]$ of massless particles. Since tree amplitudes are *rational functions* of kinematic variables(Mandelstam variables) constructed from external momenta[1], tree amplitudes are meromorphic functions of kinematic variables. The poles of tree amplitudes are determined by polology. We also know that meromorphic functions admit a pole expansion. So, how can we do a pole expansion?

The key idea of BCFW recursion is to introduce the following complex momentum shift,

$$p_i \to \hat{p}_i = p_i + zq \tag{6}$$

$$p_j \to \hat{p}_j = p_j - zq \tag{7}$$

where q is chosen to satisfy the conditions $q^2 = p_i \cdot q = p_j \cdot q = 0$ to keep shifted momenta $\hat{p}_i$ and $\hat{p}_j$ on-shell[2]. Under this shift, the n-point tree amplitude becomes a function of complex variable z, $A_n(z)$. The original amplitude $A_n[p_i]$ we wish to compute is now $A_n(0)$. Consider the following contour integral.

$$\frac{1}{2\pi i}\int_{C_\infty} \frac{dz}{z} A_n(z) = A_n(0) + \sum_I \operatorname*{Res}_{z=z_I} \frac{A_n(z)}{z} \tag{8}$$

The contour of the integral C_∞ is the circle on the complex plane with very large radius approaching infinity. Polology determines the residues on the right hand side of (8). Diagrammatically, the residues come from the contributions corresponding to the product of two on-shell scattering amplitudes with less number of external legs and a propagator[3]. This is depicted in figure 2.

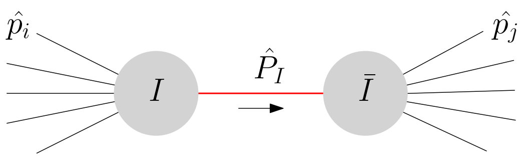

그림 2: Diagrammatic expression for (9).

The external legs are grouped into set I and its complement $\bar{I}$, where the i-th leg with shifted momentum $\hat{p}_i$ and j-th leg with shifted momentum $\hat{p}_j$ must not be in the same group. This is obvious as poles that can be surveyed by the auxiliary complex variable z are the poles coming from such diagrams. Computing the residue gives the following formula.

$$\operatorname*{Res}_{z=z_I} \frac{A_n(z)}{z} = \operatorname*{Res}_{z=z_I} A_I(z)\frac{1}{z\hat{P}_I{}^2}A_{\bar{I}}(z) = -A_I(z_I)\frac{1}{P_I^2}A_{\bar{I}}(z_I) \tag{9}$$

[1] This means tree amplitudes have interpretation as contribution from single particle intermediate states. Loop amplitudes are contributions from multi-particle intermediate states, which is naturally linked to branch cuts that appear at loop order; the branch cuts correspond to propagators for multi-particle states which have continuous on-shell spectra

[2] Such a q can be always found in dimensions $d \geq 4$. An explicit solution for $d = 4$ can be written down in spinor-helicity variables; $|\hat{i}] = |i] + z\,|j]$ and $|\hat{j}\rangle = |j\rangle - z\,|i\rangle$, while keeping other spinor-helicity variables fixed.

[3] They are sometimes called *subamplitudes*.

Going back to (8), we get the wanted amplitude as a sum over possible groupings I and a remainder coming from the integral. If the left hand side of (8) is zero, we have the following on-shell recursion relations.

$$A_n[p_i] = A_n(0) = \sum_I A_I(z_I)\frac{1}{P_I^2}A_{\bar{I}}(z_I) \tag{10}$$

This is the celebrated BCFW recursion relation. Note that we need the following constraint on large z behaviour of the amplitude for (10) to hold.

$$\frac{1}{2\pi i}\int_{C_\infty}\frac{dz}{z}A_n(z) = 0 \Longleftrightarrow A_n(z) \in \mathcal{O}(z^{-1}) \tag{11}$$

The relation (10) appeared in [2] and that the condition (11) is satisfied by Yang-Mills theory was proved in [3]. It is also known that this condition is satisfied for gravity amplitudes as well [9]. This condition will prove to be an obstruction for recursive construction of tree amplitudes for effective scalar theories.

More details and references are left to the book by Elvang and Huang[9], which is an excellent introduction to scattering amplitudes and on-shell recursion methods.

1.3 Adler's zero

Adler's zero refers to the vanishing of the amplitudes in the soft limits of Goldstone bosons. The proof given in chapter 19.2 of [10] will be reproduced here.

Consider a conserved current inserted between the vacuum state and a single Goldstone boson excitation state. Lorentz covariance forces the term to be written in the following form.

$$\langle \mathrm{vac}|\, J^\mu(x)\, |B(p)\rangle = F p_B^\mu e^{-ip_B x} \tag{12}$$

Current conservation is guaranteed by masslessness of the Goldstone bosons $p_B^2 = 0$. Now, consider the following current insertion between two asymptotic states $\langle\beta|$ and $|\alpha\rangle$.

$$\langle\beta|\, J^\mu(x)\, |\alpha\rangle = e^{-iqx}\langle\beta|\, J^\mu(0)\, |\alpha\rangle \tag{13}$$

The momentum q is defined as $q = p_\alpha - p_\beta$. Let us study the pole structure of this matrix element.

$$\langle\beta|\, J^\mu(0)\, |\alpha\rangle = \langle \mathrm{vac}|\, J^\mu(0)\, |\alpha + (\beta^*)\rangle + \cdots \tag{14}$$

$$= \langle \mathrm{vac}|\, J^\mu(0)\, |B(q)\rangle \frac{1}{q^2}\langle B(q)|\alpha + (\beta^*)\rangle + \cdots \tag{15}$$

$$= \frac{q^\mu}{q^2}F\langle \beta + B(q)|\alpha\rangle + \cdots \tag{16}$$

Define $N_{\beta\alpha}(q) = \langle\beta + B(q)|\alpha\rangle$ as the S-matrix element for emitting a Goldstone boson of momentum q during transition from the state α to the state β. Then (12) takes the following form.

$$\langle\beta|\, J^\mu(x)\, |\alpha\rangle = \frac{q^\mu}{q^2}F N_{\beta\alpha}(q)e^{-iqx} + M^\mu_{\beta\alpha}(q) \tag{17}$$

$M^{\mu}_{\beta\alpha}(q)$ represents the remaining terms that are not captured by $N_{\beta\alpha}(q)$. Current conservation implies the following.

$$\langle\beta|\, \partial_\mu J^\mu(0)\, |\alpha\rangle = 0 = F N_{\beta\alpha}(q) + q_\mu M^{\mu}_{\beta\alpha}(q) \tag{18}$$

Unless $M^{\mu}_{\beta\alpha}(q)$ has a pole at $q = 0$, we get the following vanishing soft behaviour.

$$\lim_{q\to 0} N_{\beta\alpha}(q) = \lim_{q\to 0} \langle\beta + B(q)|\alpha\rangle = 0 \tag{19}$$

This is called *Adler's zero*. One way of understanding this behaviour is to note that Goldstone bosons are related to broken symmetry generators, the symmetry generators that does not annihilate the vacuum. Addition of a Goldstone boson in the soft limit $q \to 0$ corresponds to shifting the vacuum by the corresponding broken symmetry generator. Thus, vanishing of this amplitude is a sign that different vacua do not overlap, alluding to the existence of superselection sectors.

How fast will an amplitude vanish as the momentum of a Goldstone boson approaches 0? Let us scale the momentum of a Goldstone boson by τ; $p = \tau\hat{p}$. The amplitude will have the form $A_n(\tau) \in \mathcal{O}(\tau^\sigma)$ for some positive power σ. The following are examples of theories with $\sigma = 1$, 2, and 3 [6].

σ	possible theories
1	NLSM
2	DBI, Galileon
3	sGalileon

For higher powers of σ, [6] has shown that locality forbids $\sigma \geq 4$.

2 On-shell recursion using soft behaviour

Consider a tree level scattering amplitude of Goldstone bosons. Suppose that the power of Adler's zero is fixed to σ.

$$A_n[p] \sim (p_i)^\sigma \text{ as } p_i \to 0 \tag{20}$$

Now, consider the following *all-line soft shift* [1].

$$p_i \to \hat{p}_i = p_i(1 - z a_i) \text{ for } 1 \leq i \leq n \tag{21}$$

The on-shell condition of each particle is automatically satisfied for the shifted momenta $\hat{p}_i$. Momentum conservation requires the condition

$$\sum_{i=1}^{n} a_i p_i = 0 \tag{22}$$

which has a nontrivial solution for general momenta when $n > d+1$. The n-point tree amplitude becomes a function of a complex variable z, which has the following behaviour required by (20).

$$A_n(z) \sim (1 - za_i)^\sigma \ \text{as} \ z \to \frac{1}{a_i} \tag{23}$$

Consider the following contour integral.

$$\frac{1}{2\pi i}\int_{C_\infty} \frac{A_n(z)}{zF(z)}dz = A_n(0) + \sum_I \operatorname*{Res}_{z=z_I} \frac{A_n(z)}{zF(z)} \tag{24}$$

Again, the contour of the integral C_∞ is the circle on the complex plane with very large radius approaching infinity. The function $F(z)$ is defined by the shift parameters a_i.

$$F(z) = \prod_{i=1}^{n}(1 - a_i z)^\sigma \tag{25}$$

The poles introduced by the function $F(z)$ is cancelled by the soft behaviour (20), so the residue contributions of (24) come from the poles of $A_n(z)$. Those poles have a diagrammatic representation as figure 3, analogous to BCFW recursion.

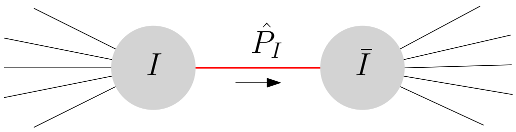

그림 3: Diagrammatic expression for (26).

Unlike BCFW recursion, the diagrammatic representation corresponds to two poles z_{I_+} and z_{I_-}. The following results are gained when the residue is computed.

$$\operatorname*{Res}_{z=z_I} \frac{A_n(z)}{zF(z)} = \operatorname*{Res}_{z=z_I} A_I(z)\frac{1}{zF(z)\hat{P}_I^{\,2}}A_{\bar{I}}(z) = -\frac{1}{P_I^2}\left[\frac{A_I(z_{I_-})A_{\bar{I}}(z_{I_-})}{(1 - z_{I_-}/z_{I_+})F(z_{I_-})} + (z_{I_-} \leftrightarrow z_{I_+})\right] \tag{26}$$

So, if the left hand side of (24) vanishes, we get the following recursion relation of on-shell amplitudes.

$$A_n[p_i] = A_n(0) = \sum_I \frac{1}{P_I^2}\left[\frac{A_I(z_{I_-})A_{\bar{I}}(z_{I_-})}{(1 - z_{I_-}/z_{I_+})F(z_{I_-})} + (z_{I_-} \leftrightarrow z_{I_+})\right] \tag{27}$$

Vanishing of the contour integral in (24) requires a nontrivial constraint. Assume large z behaviour of $A_n(z)$ as $A_n(z) \sim z^m$. The condition that the contour integral vanishes becomes

$$\frac{A_n(z)}{F(z)} \sim z^{m-n\sigma} \in \mathcal{O}(z^{-1}) \tag{28}$$

or

$$m - n\sigma < 0 \Longleftrightarrow \frac{m}{n} < \sigma. \tag{29}$$

Note that this constraint can be used to bootstrap the effective Lagrangian with the following ansatz.

$$\mathcal{L} = (\partial\phi)^2 \sum_{m',n'} \lambda_{m',n'} \partial^{m'} \phi^{n'} \tag{30}$$

The ratio $\rho = m'/n'$ is held fixed. Note that fixing this ratio fixes the large momentum behaviour of all subamplitudes to the same value. In terms of m and n of $A_n(z) \sim z^m$, ρ is given as $\rho = (m-2)/(n-2)$. Recasting (29) in terms of ρ, we get the following inequality that needs to be satisfied by ρ and σ.

$$m - 2 < n\sigma - 2 \Longrightarrow \rho < \frac{n\sigma - 2}{n-2} < \frac{\sigma - 1}{1 - 2/n} \tag{31}$$

This inequality is satisfied by theories defined as *exceptional theories* [1, 6] which exhibit an enhanced soft behaviour; for those theories the tree amplitudes vanish faster than naïve power-counting expectation from the Lagrangian in the soft limit. This enhancement is due to symmetries that relate low-order contact operators to those of higher order [1].

Fixing ρ and σ gives a classification of effective scalar theories known as (ρ, σ) classification [11], and [6] attempted to exhaust all possible effective theories based on this classification. Some of the nontrivial exceptional theories found in [6] is reproduced below.

(ρ, σ)	possible theories
$(0, 1)$	NLSM
$(1, 2)$	DBI
$(2, 2)$	Galileon
$(2, 3)$	sGalileon

Momentum shifts other than (21) that can be used to derive recursive relations of on-shell tree amplitudes can be found in [6]. Example computations using this recursive structure of on-shell tree amplitudes can be found in [1].

3 Summary

The findings of [1] were reviewed which states that soft behaviour can be used to derive recursive structures of tree amplitudes for effective scalar theories. It would be interesting to see whether this new method can be used to derive more simple recursive structures for scattering amplitudes of massive particles.

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